Stochastic Processes with Balking in the Theory of Telephone Traffic*

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It is supposed that at a telephone exchange calls are arriving according to a recurrent process. If an incoming call finds exactly j lines busy then it either realizes a connection with probability p_j or balks with probability q_j ($p_j + q_j = 1$). The holding times are mutually independent random variables with common exponential distribution. In this paper the stochastic behavior of the fluctuation of the number of the busy lines is studied.

I. INTRODUCTION

Many results in telephone traffic theory (and elsewhere) may be unified by the introduction of *balking*. A call is said to balk if for any reason it refuses service on arrival. A mathematical model for balking is constructed by assigning a probability to balking dependent only on the state of the system; if an incoming call finds exactly j lines busy, then it realizes a connection with probability p_j and balks with probability q_j ($p_j + q_j = 1$). Thus if $p_j = 1$ ($j = 0, 1, \cdots$) the system is one with an infinite number of lines and with no loss and no delay, the ideal for for any service, while if $p_j = 1$ ($j = 0, 1, \cdots, m - 1$) and $p_j = 0$ ($j = m, m + 1, \cdots$) the system is a loss system with m lines.

This balking model is examined here for recurrent input and exponential distribution of holding times. More specifically, the call arrival times are taken as the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$, where the interarrival times $\theta_n = \tau_{n+1} - \tau_n$ ($n = 0, 1, \cdots; \tau_0 = 0$) are identically distributed, mutually independent, positive random variables with distribution function

$$\mathbf{P}\{\theta_n \leqq x\} = F(x). \tag{1}$$

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* Dedicated to the memory of my professor Charles Jordan (December 16, 1871–December 24, 1959)

The holding times are identically distributed, mutually independent random variables with distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{2}$$

The holding times are independent of the $\{\tau_n\}$ as well.

Let us denote by $\xi(t)$ the number of busy lines at the instant t. Define $\xi_n = \xi(\tau_n - 0)$; that is, ξ_n is the number of busy lines immediately before the arrival of the nth call. The system is said to be in state E_k at the instant t if $\xi(t) = k$. Let us denote by m the smallest nonnegative integer such that $p_m = 0$. If $p_j > 0$ $(j = 0, 1, 2, \cdots)$ then $m = \infty$.

In the present paper we shall give a method to determine the distribution of ξ_n for every n, the distribution of $\xi(t)$ for finite t values, and the limiting distributions of ξ_n and $\xi(t)$ as $n \to \infty$ and $t \to \infty$ respectively. Further, we shall determine the stochastic law of the transitions $E_k \to E_{k+1}$ $(k = 0, 1, 2, \cdots)$.

II. NOTATION

The Laplace-Stieltjes transform of the distribution function of the interarrival times will be denoted by

$$\varphi(s) = \mathbf{E}\{e^{-s\theta_n}\} = \int_0^\infty e^{-sx}\, dF(x),$$

which is convergent if $\Re(s) \geqq 0$. The expectation of the interarrival times will be denoted by

$$\alpha = \mathbf{E}\{\theta_n\} = \int_0^\infty x\, dF(x).$$

Let $\mathbf{P}\{\xi_n = k\} = P_k^{(n)}$ and $\mathbf{P}\{\xi(t) = k\} = P_k(t)$. Define

$$\Pi_k(s) = \int_0^\infty e^{-st} P_k(t)\, dt,$$

which is convergent if $\Re(s) > 0$. Let

$$\lim_{n\to\infty} P_k^{(n)} = P_k \qquad \text{and} \qquad \lim_{t\to\infty} P_k(t) = P_k^*,$$

provided that the limits exist.

Define

$$C_r = \prod_{i=1}^{r} \left(\frac{\varphi(i\mu)}{1 - \varphi(i\mu)}\right) \qquad (r = 0, 1, 2, \cdots), \tag{3}$$

where the empty product means 1; that is, $C_0 = 1$. We shall also use the abbreviation

$$\varphi_r = \varphi(r\mu) = \int_0^\infty e^{-r\mu x}\, dF(x) \qquad (r = 0, 1, 2, \cdots). \tag{4}$$

Denote by $M_k(t)$ the expected number of calls occurring in the time interval $(0,t]$ which find exactly k lines busy. The expected number of transitions $E_k \rightarrow E_{k+1}$ occurring in tbe time interval $(0,t]$ is clearly $p_k M_k(t)$. Denote by $N_k(t)$ the expected number of transitions $E_{k+1} \rightarrow E_k$ occurring in the time interval $(0,t]$.

Let $G_k(x)(k = 0, 1, 2, \cdots)$ be the distribution function of the time differences between successive transitions $E_{k-1} \rightarrow E_k$ and $E_k \rightarrow E_{k+1}$, while $R_k(x)(k = 0, 1, 2, \cdots)$ is the distribution function of the time differences between consecutive transitions $E_k \rightarrow E_{k+1}$. If $\xi(0) = 0$ then we say that a transition $E_{-1} \rightarrow E_0$ takes place at time $t = -0$. Write

$$\gamma_k(s) = \int_0^\infty e^{-sx}\, dG_k(x)$$

and

$$\rho_k(s) = \int_0^\infty e^{-sx}\, dR_k(x)$$

which are convergent if $\Re(s) \geqq 0$.

III. PREVIOUS RESULTS

3.1 *A. K. Erlang*

Erlang[1] has proved that, if $\{\tau_n\}$ forms a Poisson process of intensity λ—that is, $F(x) = 1 - e^{-\lambda x}$ for $x \geqq 0$—and further, $p_j = 1$ when $j < m$, $p_j = 0$ when $j \geqq m$, then

$$P_k{}^* = \frac{\dfrac{(\lambda/\mu)^k}{k!}}{\displaystyle\sum_{j=0}^{m} \frac{(\lambda/\mu)^j}{j!}} \qquad (k = 0, 1, \cdots, m). \tag{5}$$

In this case $P_k = P_k{}^*$ $(k = 0, 1, \cdots, m)$ also holds. This is the simplest loss system.

3.2 *Conny Palm*

Palm[2] has generalized the above result of Erlang for the case when $\{\tau_n\}$ forms a recurrent process and otherwise every assumption remains unchanged. Palm has proved that

$$P_m = \frac{1}{\sum_{r=0}^{m} \binom{m}{r} \frac{1}{C_r}}, \tag{6}$$

where C_r is defined by (3). In this case the complete limiting distributions $\{P_k\}$ and $\{P_k^*\}$ have been determined by Pollaczek,[3] Cohen,[4] and the author.[5,6] The transient behavior of the sequence $\{\xi_n\}$ was determined by Pollaczek[3] and Beneš,[7] and the transient behavior of the process $\{\xi(t)\}$ by Beneš[8] and by the author.[9]

3.3 *The Infinite Line Case*

The case when $p_j = 1$ $(j = 0, 1, 2, \cdots)$ has been investigated by the author,[10,11] who has proved that

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r \qquad (k = 0, 1, 2, \cdots) \tag{7}$$

and, if $F(x)$ is not a lattice distribution and if $\alpha < \infty$, then the limiting distribution $\{P_k^*\}$ exists and

$$\begin{aligned} P_k^* &= \frac{P_{k-1}}{k\alpha\mu} \qquad (k = 1, 2, \cdots), \\ P_0^* &= 1 - \frac{1}{\alpha\mu} \sum_{k=1}^{\infty} \frac{P_{k-1}}{k}. \end{aligned} \tag{8}$$

The transient behavior of the process $\{\xi(t)\}$ is also treated in Refs. 10 and 11.

3.4 *The Case* $p_0 = 1$, $p_j = p$ $(j = 1, 2, \cdots)$, $q_j = q$ $(j = 1, 2, \cdots)$

This case, where $p + q = 1$, plays an important role in the theory of particle counters and has been investigated by the author,[12] who has found that

$$P_0 = \frac{p \sum_{r=0}^{\infty} (-p)^r C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \tag{9}$$

and

$$P_k = \frac{\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \qquad (k = 1, 2, \cdots). \tag{10}$$

If $F(x)$ is not a lattice distribution and if $\alpha < \infty$, then the limiting distribution $\{P_k^*\}$ exists and

$$\begin{aligned} P_{k+1}^* &= \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots), \\ P_0^* &= 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{(k+1)}. \end{aligned} \tag{11}$$

The transient behavior of the process $\{\xi(t)\}$ is also treated in Ref. 12.

3.5 *The Distribution Function $G_k(x)$*

This function plays an important role in the investigation of overflow traffic. In the infinite line case, i.e., when $p_j = 1$ $(j = 0, 1, 2, \cdots)$, Palm[2] has proved that $\gamma_k(s)(k = 0, 1, 2, \cdots)$ satisfies the following recurrence formula:

$$\gamma_k(s) = \frac{\gamma_{k-1}(s+\mu)}{1 - \gamma_{k-1}(s) + \gamma_{k-1}(s+\mu)} \qquad (k = 1, 2, \cdots), \tag{12}$$

where $\gamma_0(s) = \varphi(s)$. Palm has obtained $\gamma_k(s)$ explicitly when $\{\tau_n\}$ is a Poisson process; that is, $\varphi(s) = \lambda/(\lambda + s)$. Then

$$\gamma_k(s) = \frac{\sum_{j=0}^{k} \binom{k}{j} \dfrac{s(s+\mu)\cdots[s+(j-1)\mu]}{\lambda^j}}{\sum_{j=0}^{k+1} \binom{k+1}{j} \dfrac{s(s+\mu)\cdots[s+(j-1)\mu]}{\lambda^j}}. \tag{13}$$

The general solution of the recurrence formula (12) is

$$\gamma_k(s) = \frac{\sum_{r=0}^{k} \binom{k}{r} \prod_{i=0}^{r-1} \left[\dfrac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right]}{\sum_{r=0}^{k+1} \binom{k+1}{r} \prod_{i=0}^{r-1} \left[\dfrac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right]} \qquad (k = 0, 1, 2, \cdots), \tag{14}$$

where the empty product means 1. The formula (14) is proved in Refs. 10 and 11.

In the particular case $p_0 = 1$, $p_j = p$ $(j = 1, 2, \cdots)$, $q_j =$

$q(j = 1, 2, \cdots)$, where $p + q = 1$, the Laplace-Stieltjes transform $\gamma_k(s)$ has been given explicitly in Ref. 12. We have

$$\gamma_k(s) = \frac{D_k(s)}{D_{k+1}(s)} \qquad (k = 0, 1, 2, \cdots), \tag{15}$$

where $D_0(s) \equiv 1$ and

$$\begin{aligned} D_k(s) = \Bigg\{ p \sum_{r=0}^{k} \binom{k}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \\ - \frac{q[1 - \varphi(s)]}{p\varphi(s)} \sum_{r=0}^{k} \binom{k}{r} \sum_{j=1}^{r-1} (-1)^j \prod_{i=j+1}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \Bigg\} \end{aligned} \tag{16}$$

if $k = 1, 2, \cdots$.

IV. THE TRANSIENT BEHAVIOR OF $\{\xi_n\}$

It is easy to see that the sequence of random variables $\{\xi_n\}$ forms a homogeneous Markov chain with transition probabilities

$$p_{jk} = \mathbf{P}\{\xi_{n+1} = k \mid \xi_n = j\} = \int_0^\infty \pi_{jk}(x)\, dF(x), \tag{17}$$

where

$$\pi_{jk}(x) = p_j \binom{j+1}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j+1-k} + q_j \binom{j}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j-k} \tag{18}$$

is the conditional transition probability given that the interarrival time $\theta_n = x$ (constant). For, if $\xi_n = j$ and $\theta_n = x$, then ξ_{n+1} has a Bernoulli distribution, either with parameters $j + 1$ and $e^{-\mu x}$ when the nth call realizes a connection, or with parameters j and $e^{-\mu x}$ when the nth call does not. The system is said to be in state E_k at the nth step if $\xi_n = k$.

Starting from the initial distribution $\{P_k^{(1)}\}$ the distributions $\{P_k^{(n)}\}$ can be determined successively by the following formulas:

$$P_k^{(n+1)} = \sum_{j=k-1}^{\infty} p_{jk} P_j^{(n)} \qquad (n = 1, 2, \cdots). \tag{19}$$

However, it turns out that in many cases it is more convenient to determine the binomial moments of $\{P_k^{(n)}\}$ first. By definition,

$$U_r^{(n)} = \mathbf{E}\left\{\binom{\xi_n}{r}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} P_k^{(n)} \qquad (r = 0, 1, 2, \cdots) \tag{20}$$

is the rth binomial moment of $\{P_k^{(n)}\}$. If we suppose that $U_r^{(1)} < C_1^r/r!$ where C_1 is a constant, then it can be proved that every $U_r^{(n)}$ exists and $U_r^{(n)} < C^r/r!$ where C is a constant. Thus the distribution $\{P_k^{(n)}\}$ is uniquely determined by $\{U_r^{(n)}\}$. We obtain from (20) that

$$P_k^{(n)} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} U_r^{(n)} \qquad (k = 0, 1, 2, \cdots). \tag{21}$$

This is the inversion formula of Jordan.[13]

It is convenient to use the related quantities

$$V_r^{(n)} = \mathbf{E}\left\{\binom{\xi_n}{r} p_{\xi_n}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} p_k P_k^{(n)} \qquad (r = 0, 1, 2, \cdots), \tag{22}$$

whence by inversion

$$p_k P_k^{(n)} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} V_r^{(n)}. \tag{23}$$

Now we shall prove

Theorem 1. We have $U_0^{(n)} = 1$ $(n = 1, 2, \cdots)$ *and*

$$U_r^{(n+1)} = \varphi_r(U_r^{(n)} + V_{r-1}^{(n)}) \qquad (n = 1, 2, \cdots; \quad r = 1, 2, \cdots), \tag{24}$$

where $\varphi_r = \varphi(r\mu)$. *Further*

$$V_r^{(n)} = \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j^{(n)} \qquad (r = 0, 1, 2, \cdots), \tag{25}$$

where

$$\Delta^{j-r} p_r = \sum_{\nu=0}^{j-r} (-1)^{\nu} \binom{j-r}{\nu} p_{j-\nu}. \tag{26}$$

Proof. First of all we note that the rth binomial moment of the Bernoulli distribution $\{Q_k\}$ with parameters n and p, that is, that of

$$Q_k = \binom{n}{k} p^k (1 - p)^{n-k} \qquad (k = 0, 1, \cdots, n),$$

is given by

$$B_r = \sum_{k=r}^{n} \binom{k}{r} Q_k = \binom{n}{r} p^r \qquad (r = 0, 1, \cdots, n). \tag{27}$$

Using (27), we get by (18) that

$$\mathbf{E}\left\{\binom{\xi_{n+1}}{r} \mid \xi_n = j, \theta_n = x\right\} = p_j \binom{j+1}{r} e^{-r\mu x} + q_j \binom{j}{r} e^{-r\mu x},$$

whence

$$\begin{aligned}\mathbf{E}\left\{\binom{\xi_{n+1}}{r} \mid \xi_n = j\right\} &= \varphi_r\left[p_j\binom{j+1}{r} + q_j\binom{j}{r}\right] \\ &= \varphi_r\left[\binom{j}{r} + p_j\binom{j}{r-1}\right].\end{aligned} \tag{28}$$

If we multiply both sides of (28) by $P_j^{(n)}$ and add them for every j, then we get (24). We obtain (25) if we put (21) into (22). This completes the proof of the theorem.

Starting from $U_r^{(1)}(r = 1, 2, \cdots)$ the binomial moments $U_r^{(n)}(n = 2, 3, \cdots)$ can be obtained recursively by (24) and (25). If, specifically, $\xi(0) = i$ and $\tau_1 = x$ then ξ_1 has a Bernoulli distribution with parameters i and $e^{-\mu x}$ and thus, for $\xi(0) = i$,

$$U_r^{(1)} = \mathbf{E}\left\{\binom{\xi_1}{r}\right\} = \binom{i}{r}\varphi_r \qquad (r = 0, 1, 2, \cdots). \tag{29}$$

Remark 1. If we introduce the generating functions

$$U_r(w) = \sum_{n=1}^{\infty} U_r^{(n)} w^n \tag{30}$$

and

$$V_r(w) = \sum_{n=1}^{\infty} V_r^{(n)} w^n \tag{31}$$

and suppose that $\xi(0) = i$, then by (24) and (29) we get that

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + V_{r-1}(w)\right] \qquad (r = 1, 2, \cdots), \tag{32}$$

and evidently

$$U_0(w) = \frac{w}{1 - w}. \tag{33}$$

Note also that (21) implies that

$$\sum_{n=1}^{\infty} P_k^{(n)} w^n = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k} U_r(w). \tag{34}$$

Example 1. In the infinite line case, i.e., when $p_j = 1 (j = 0, 1, 2, \cdots)$, $V_r^{(n)} = U_r^{(n)}$ and $V_r(w) = U_r(w)$ for $r = 0, 1, 2, \cdots$. If we suppose that $\xi(0) = i$, then by (32) we get

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + U_{r-1}(w)\right] \qquad (r = 1, 2, \cdots) \tag{35}$$

and $U_0(w) = w/(1 - w)$. The solution of these equations is given by

$$U_r(w) = \left\{\prod_{j=0}^{r}\left(\frac{w\varphi_j}{1 - w\varphi_j}\right)\right\}\left\{\sum_{j=0}^{r}\binom{i}{j}\prod_{\nu=0}^{j-1}\left(\frac{1 - w\varphi_\nu}{w\varphi_\nu}\right)\right\} \qquad (r = 0, 1, 2, \cdots),$$

where the empty product means 1. The distribution $\{P_k^{(n)}\}$ is determined by (34).

Example 2. For a loss system with m lines, i.e., when $p_j = 1$ $(j < m)$ and $p_j = 0$ $(j \geqq m)$, in the case $\xi(0) = i \leqq m$ we have

$$V_r^{(n)} = U_r^{(n)} - \binom{m}{r} U_m^{(n)} \qquad (r = 0, 1, 2, \cdots, m - 1)$$

and

$$V_r^{(n)} = U_r^{(n)} = 0 \qquad (r = m, m + 1, \cdots).$$

Thus,

$$V_r(w) = U_r(w) - \binom{m}{r} U_m(w) \qquad (r = 0, 1, 2, \cdots, m - 1)$$

and

$$V_r(w) = U_r(w) = 0 \qquad (r = m, m + 1, \cdots).$$

By (32) we get

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + U_{r-1}(w) - \binom{m}{r - 1} U_m(w)\right] \tag{36}$$

$$(r = 1, 2, \cdots, m)$$

and $U_0(w) = w/(1 - w)$. The solution of these equations for $r = 0, 1, 2, \cdots, m$ is given by

$$U_r(w) = \frac{\Gamma_r(w)}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{\Gamma_j(w)}}\left\{\left[\sum_{j=r}^{m}\binom{m}{j}\frac{1}{\Gamma_j(w)}\right]\left[\sum_{j=0}^{r}\binom{i}{j}\frac{1}{\Gamma_{j-1}(w)}\right]\right. \tag{37}$$

$$\left. - \left[\sum_{j=0}^{r-1}\binom{m}{j}\frac{1}{\Gamma_j(w)}\right]\left[\sum_{j=r+1}^{m}\binom{i}{j}\frac{1}{\Gamma_{j-1}(w)}\right]\right\}$$

where

$$\Gamma_r(w) = \prod_{i=0}^{r}\left(\frac{w\varphi_i}{1 - w\varphi_i}\right), \qquad (r = 0, 1, 2, \cdots)$$

and $\Gamma_{-1}(w) \equiv 1$. Finally, $\{P_k^{(n)}\}$ can be obtained by (34).

V. THE LIMITING DISTRIBUTION $\{P_k\}$

In the Markov chain $\{\xi_n\}$ the states E_0, E_1, $\cdots$, E_m form an irreducible closed set, while E_m, E_{m+1}, $\cdots$ are transient states. If either $m = \infty$ or $m < \infty$, but we restrict ourselves to the states E_0, E_1, $\cdots$, E_m, then the Markov chain $\{\xi_n\}$ is irreducible. The Markov chain $\{\xi_n\}$ is always aperiodic. Accordingly

$$\lim_{n\to\infty} P_k^{(n)} = P_k \qquad (k = 0, 1, 2, \cdots)$$

always exists and is independent of the initial distribution. There are two possibilities: either every $P_k = 0$ $(k = 0, 1, 2, \cdots)$ or $\{P_k\}$ is a probability distribution. (In the second case $P_k > 0$ if $k \leqq m$ and $P_k = 0$ if $k > m$.) In the second case $\{P_k\}$ is the unique stationary distribution of the Markov chain $\{\xi_n\}$ and conversely if there exists a stationary distribution then it is unique and agrees with the limiting distribution $\{P_k\}$.

In the particular case $p_j = 1$ $(j = 0, 1, 2, \cdots)$ the limiting distribution always exists, as has been proved in Ref. 10. In this special case

$$P_0 = \sum_{r=0}^{\infty} (-1)^r C_r > 0.$$

If we consider an arbitrary sequence $\{p_j\}$ then evidently

$$P_0 \geqq \sum_{r=0}^{\infty} (-1)^r C_r > 0,$$

whence it follows that $\{\xi_n\}$ belongs to the second class; that is, $\{P_k\}$ is a probability distribution.

The stationary distribution $\{P_k\}$ is uniquely determined by the following system of linear equations:

$$P_k = \sum_{j=k-1}^{\infty} p_{jk} P_j \tag{38}$$

and

$$\sum_{k=1}^{\infty} P_k = 1. \tag{39}$$

Since in this case $P_k^{(n)} = P_k$ for every n, we get (38) by (19). Now let us introduce the binomial moments

$$U_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k \qquad (r = 0, 1, 2, \cdots) \tag{40}$$

and define

$$V_r = \sum_{k=r}^{\infty} \binom{k}{r} p_k P_k . \tag{41}$$

By inversion we get, from (40),

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} U_r \qquad (k = 0, 1, 2, \cdots) \tag{42}$$

and similarly, from (41),

$$p_k P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} V_r . \tag{43}$$

The binomial moments $U_r (r = 0, 1, 2, \cdots)$ can be obtained by the following

Theorem 2. We have $U_0 = 1$ *and*

$$U_r = \frac{\varphi_r}{1 - \varphi_r} V_{r-1} \qquad (r = 1, 2, \cdots), \tag{44}$$

where $\varphi_r = \varphi(r\mu)$. *Further,*

$$V_r = \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j \qquad (r = 0, 1, 2, \cdots), \tag{45}$$

where

$$\Delta^{j-r} p_r = \sum_{\nu=0}^{j-r} (-1)^{\nu} \binom{j-r}{\nu} p_{j-\nu} . \tag{46}$$

Proof. This theorem immediately follows from Theorem 1 if we put $U_r^{(n)} = U_r$, $V_r^{(n)} = V_r$ in (24) and (25).

Remark 2. In many cases there is a simple relation between the generating functions

$$U(z) = \sum_{k=0}^{\infty} P_k z^k \tag{47}$$

and

$$V(z) = \sum_{k=0}^{\infty} p_k P_k z^k \tag{48}$$

when U_r $(r = 0, 1, \cdots)$ can easily be obtained by (44). For,

$$U_r = \frac{1}{r!} \left[\frac{d^r U(z)}{dz^r} \right]_{z=1} \qquad (r = 0, 1, 2, \cdots) \tag{49}$$

and

$$V_r = \frac{1}{r!}\left[\frac{d^r V(z)}{dz^r}\right]_{z=1} \qquad (r = 0, 1, 2, \cdots). \tag{50}$$

Theorem 3. The binomial moments U_r $(r = 0, 1, 2, \cdots)$ *satisfy the following system of linear equations:*

$$\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k}\left(p_k U_r - \frac{1 - \varphi_{r+1}}{\varphi_{r+1}} U_{r+1}\right) = 0 \qquad (r = 0, 1, 2, \cdots) \tag{51}$$

and

$$U_{r+1} = \frac{\varphi_{r+1}}{1 - \varphi_{r+1}} \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j \qquad (r = 0, 1, 2, \cdots), \tag{52}$$

where $\Delta^{j-r} p_r$ *is defined by* (46).

Proof. If we put (42) into (43) and use the relation (44) then we get (51). If we eliminate V_r from (44) and (45) then we get (52).

Remark 3. If $p_m = 0$ then $U_r = 0$ for $r > m$, and in this case, starting from U_m, the unknowns $U_{m-1}, U_{m-2}, \cdots, U_0$ can be obtained successively either by (51) or by (52) and finally $U_0 = 1$ determines U_m. If the higher differences of p_r vanish, then (52) can be used successfully for the determination of the binomial moments U_r.

Example 3. If $p_j = 1$ $(j = 0, 1, 2, \cdots)$ then $V_r = U_r$ $(r = 0, 1, 2, \cdots)$ and, by (44),

$$U_r = \frac{\varphi_r}{1 - \varphi_r} U_{r-1} \qquad (r = 1, 2, \cdots),$$

whence

$$U_r = \prod_{j=1}^{r} \left(\frac{\varphi_j}{1 - \varphi_j}\right) \qquad (r = 1, 2, \cdots) \tag{53}$$

and $U_0 = 1$. The distribution $\{P_k\}$ is given by (42).

Example 4. Let $p_j = 1$ if $j < m$ and $p_j = 0$ if $j \geqq m$. Then

$$V_r = U_r - \binom{m}{r} U_m \qquad (r = 0, 1, \cdots, m)$$

and

$$V_r = U_r = 0 \qquad (r = m + 1, m + 2, \cdots).$$

By (44)

$$U_r = \frac{\varphi_r}{1 - \varphi_r}\left[U_{r-1} - \binom{m}{r-1} U_m\right] \qquad (r = 1, 2, \cdots, m),$$

and the solution of this equation is

$$U_r = C_r \frac{\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j}} \qquad (r = 0, 1, \cdots, m), \tag{54}$$

where C_r is defined by (3). $U_r = 0$ if $r > m$. Finally, $\{P_k\}$ is given by (42).

Example 5. Let $p_0 = 1$ and $p_j = p$ $(j = 1, 2, \cdots)$, $q_j = q$ $(j = 1, 2, \cdots)$, where $p + q = 1$. Then

$$V_r = pU_r \qquad (r = 1, 2, \cdots),$$

$$V_0 = pU_0 + qP_0 = 1 - q\,(U_1 - U_2 + U_3 - \cdots).$$

Putting V_r into (44) we get

$$U_r = \frac{p\varphi_r}{1 - \varphi_r} U_{r-1} \qquad (r = 1, 2, \cdots)$$

and

$$U_r = \frac{\varphi_1}{1 - \varphi_1}[1 - q(U_1 - U_2 + U_3 - \cdots)].$$

The solution of this system of linear equations is

$$U_r = \frac{p^r C_r}{1 - q \sum_{j=0}^{\infty} (-p)^j C_j} \qquad (r = 0, 1, 2, \cdots), \tag{55}$$

where C_r is defined by (3). Finally, $\{P_k\}$ is given by (42).

Example 6. Let $p_0 = 1$, $p_j = p$, and $q_j = q$ if $j = 1, 2, \cdots, m - 1$, where $p + q = 1$, and $p_j = 0$ if $j > m$. Then

$$\begin{aligned} V_0 &= p + qP_0 - pP_m \\ &= p + q[U_0 - U_1 + U_2 - \cdots + (-1)^m U_m] - pU_m, \end{aligned}$$

$$V_r = pU_r - p\binom{m}{r} U_m \qquad (r = 1, 2, \cdots, m),$$

$$V_r = U_r = 0 \qquad (r = m + 1, m + 2, \cdots).$$

Now $U_0 = 1$ and, by (44),

$$U_r = \frac{p^r C_r \sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j p^j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j p^j} - q \sum_{j=0}^{m} (-1)^j C_j p^j \sum_{i=j}^{m} \binom{m}{i} \frac{1}{C_i p^i}} \tag{56}$$

$$(r = 1, 2, \cdots, m).$$

The distribution $\{P_k\}$ is given by (42).

Example 7. If, in particular, $F(x) = 1 - e^{-\lambda x}$ for $x \geqq 0$, then $\varphi(s) = \lambda/(\lambda + s)$ and $\varphi_r = \lambda/(\lambda + r\mu)(r = 0, 1, 2, \cdots)$. In this case by (24) we have

$$r\mu U_r = \lambda V_{r-1} \qquad (r = 1, 2, \cdots),$$

whence

$$\mu U'(z) = \lambda V(z).$$

Forming the coefficient of z^{k-1} we obtain that

$$\mu k P_k = \lambda p_{k-1} P_{k-1} \qquad (k = 1, 2, \cdots), \tag{57}$$

whence

$$P_k = P_0 \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} p_0 p_1 \cdots p_k \qquad (k = 0, 1, 2, \cdots),$$

and P_0 is determined by the requirement that

$$\sum_{k=0}^{\infty} P_k = 1.$$

VI. THE TRANSIENT BEHAVIOR OF $\{\xi(t)\}$

In this section we suppose that $\xi(0) = i$ always. Denote by $M_j(t)$ the expectation of the number of calls occurring in the time interval $(0,t]$ which find exactly j lines busy. Let

$$\mu_j(s) = \int_0^{\infty} e^{-st}\, dM_j(t), \tag{58}$$

which is convergent if $\Re(s) > 0$. Now we shall prove the following

Lemma 1. Define

$$\Phi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} \mu_j(s) \qquad (r = 0, 1, 2, \cdots) \tag{59}$$

and

$$\Psi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} p_j \mu_j(s) \qquad (r = 0, 1, 2, \cdots), \tag{60}$$

which are convergent if $\Re(s) > 0$. *Then*

$$\Phi_0(s) = \frac{\varphi(s)}{1 - \varphi(s)} \tag{61}$$

and if $\xi(0) = i$ *then*

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{1 - \varphi(s + r\mu)} \left[\binom{i}{r} + \Psi_{r-1}(s)\right]. \tag{62}$$

Proof. Since evidently

$$M_j(t) = \sum_{n=1}^{\infty} \mathbf{P}\{\tau_n \leqq t, \quad \xi_n = j\}, \tag{63}$$

we have

$$\Phi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} \mu_j(s) = \sum_{n=1}^{\infty} \mathbf{E}\left\{e^{-s\tau_n} \binom{\xi_n}{r}\right\} \tag{64}$$

and similarly

$$\Psi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} p_j \mu_j(s) = \sum_{n=1}^{\infty} \mathbf{E}\left\{e^{-s\tau_n} \binom{\xi_n}{r} p_{\xi_n}\right\}. \tag{65}$$

Now we shall prove that

$$\mathbf{E}\left\{e^{-s\tau_{n+1}} \binom{\xi_{n+1}}{r} \mid \xi_n = j, \quad \theta_n = x, \quad \tau_n = y\right\}$$
$$= \left[p_j \binom{j+1}{r} + q_j \binom{j}{r}\right] e^{-r\mu x} e^{-s(x+y)}.$$

This follows from the fact that under the given condition ξ_{n+1} has a Bernoulli distribution either with parameters $j + 1$ and $e^{-\mu x}$ when the nth call gives rise to a connection, or with parameters j and $e^{-\mu x}$ when the nth call does not. Unconditionally we get

$$\mathbf{E}\left\{e^{-s\tau_{n+1}} \binom{\xi_{n+1}}{r}\right\}$$
$$= \varphi(s + r\mu) \left[\mathbf{E}\left\{e^{-s\tau_n} \binom{\xi_n}{r}\right\} + \mathbf{E}\left\{e^{-s\tau_n} \binom{\xi_n}{r-1} p_{\xi_n}\right\}\right]. \tag{66}$$

If $\xi(0) = i$ then

$$\mathbf{E}\left\{e^{-s\tau_1}\binom{\xi_1}{r}\right\} = \binom{i}{r}\varphi(s + r\mu). \tag{67}$$

If we add (66) for $n = 1, 2, \cdots$ and (67) then we get

$$\Phi_r(s) = \varphi(s + r\mu)\left[\binom{i}{r} + \Phi_r(s) + \Psi_{r-1}(s)\right] \qquad (r = 0, 1, 2, \cdots), \tag{68}$$

where $\Psi_{-1}(s) \equiv 0$. Thus we get (61) and (62). In many cases use of Lemma 1 determines $\Phi_r(s)$ $(r = 0, 1, 2, \cdots)$ explicitly.

Remark 4. From (59) we obtain by inversion

$$\mu_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k}\Phi_r(s). \tag{69}$$

The functions $\mu_k(s)(k = 0, 1, 2, \cdots)$ can be determined also by the following system of linear equations:

$$\sum_{k=r}^{\infty}\binom{k}{r}\mu_k(s) = \frac{\varphi(s + r\mu)}{1 - \varphi(s + r\mu)}\left[\binom{i}{r} + \sum_{k=r-1}^{\infty}\binom{k}{r-1}p_k\mu_k(s)\right], \tag{70}$$

which we get if we put (59) and (60) into (62).

If we know $\Phi_r(s)$ $(r = 0, 1, 2, \cdots)$ then $P_k(t)$ can be determined by the following

Theorem 4. The Laplace transform $\Pi_k(s)$ is given by

$$\Pi_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k}\beta_r(s), \tag{71}$$

where

$$\beta_r(s) = \frac{[1 - \varphi(s + r\mu)]\Phi_r(s)}{\varphi(s + r\mu)(s + r\mu)} \qquad (r = 0, 1, 2, \cdots). \tag{72}$$

Proof. Let the rth binomial moment of $\{P_k(t)\}$ be defined by

$$B_r(t) = \mathbf{E}\left\{\binom{\xi(t)}{r}\right\} = \sum_{k=r}^{\infty}\binom{k}{r}P_k(t) \qquad (r = 0, 1, 2, \cdots). \tag{73}$$

By using the results of Ref. 10 we can see that $B_r(t) \leqq C^r/r!$ for every $t \geqq 0$, where C is a constant. Thus the probability distribution $\{P_k(t)\}$ is uniquely determined by its binomial moments. From (73) we get by inversion

$$P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k}B_r(t). \tag{74}$$

If

$$\beta_r(s) = \int_0^\infty e^{-st} B_r(t)\, dt$$

and we form the Laplace transform of (74), we get (71). Now let us determine $\beta_r(s)$ $(r = 0, 1, 2, \cdots)$.

If $\xi(0) = i$, then

$$\begin{aligned} B_r(t) &= \binom{i}{r} e^{-r\mu t}[1 - F(t)] \\ &+ \sum_{j=0}^{\infty} \left[p_j \binom{j+1}{r} + q_j \binom{j}{r} \right] \int_0^t e^{-r\mu(t-u)}[1 - F(t-u)]\, dM_j(u), \end{aligned} \tag{75}$$

where $M_j(t)$ is defined by (63). For, if there is no call in the time interval $(0,t]$ then $\xi(t)$ has a Bernoulli distribution with parameters i and $e^{-\mu t}$. If the last call in the time interval $(0,t]$ occurs at the instant u and in that instant the number of busy lines is j, then $\xi(t)$ has a Bernoulli distribution, either with parameters $j + 1$ and $e^{-\mu(t-\mu)}$ when this call gives rise to a connection or with parameters j and $e^{-\mu(t-\mu)}$ when this call does not. If we also take into consideration that the last call occurring in the time interval $(0,t]$ may be the 1st, 2nd, $\cdots$, nth, $\cdots$ one, then we get (75). Forming the Laplace transform of (74) we get

$$\begin{aligned} \beta_r(s) = \frac{1 - \varphi(s + r\mu)}{s + r\mu} \Bigg\{ &\binom{i}{r} \\ &+ \sum_{j=0}^{\infty} \left[p_j \binom{j+1}{r} + q_j \binom{j}{r} \right] \mu_j(s) \Bigg\} \end{aligned} \tag{76}$$

where $\mu_j(s)$ is defined by (58). By using the notations (59) and (60) we can write also that

$$\beta_r(s) = \frac{1 - \varphi(s + r\mu)}{s + r\mu} \left\{ \binom{i}{r} + \Phi_r(s) + \Psi_{r-1}(s) \right\}. \tag{77}$$

Taking into consideration the relation (68) we obtain finally

$$\beta_r(s) = \frac{1 - \varphi(s + r\mu)}{(s + r\mu)} \frac{\Phi_r(s)}{\varphi(s + r\mu)}, \tag{78}$$

which was to be proved.

Example 8. Define

$$C_r(s) = \prod_{i=0}^{r} \left(\frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)} \right) \quad (r = 0, 1, 2, \cdots) \tag{79}$$

and

$$C_{-1}(s) \equiv 1.$$

If $p_j = 1$ ($j = 0, 1, 2, \cdots$) and $\xi(0) = i$, then $\Psi_r(s) = \Phi_r(s)$ ($r = 0, 1, 2, \cdots$) and, by (62), we get

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{[1 - \varphi(s + r\mu)]} \left[\binom{i}{r} + \Phi_{r-1}(s)\right] \qquad (r = 0, 1, \cdots), \tag{80}$$

where $\Phi_{-1}(s) = 0$. The solution of this recurrence formula is

$$\Phi_r(s) = C_r(s) \sum_{j=0}^{r} \binom{i}{j} \frac{1}{C_{j-1}(s)}, \tag{81}$$

where $C_r(s)$ is defined by (79).

Example 9. If $p_j = 1$ when $j < m$ and $p_j = 0$ when $j \geqq m$ and $\xi(0) = i \leqq m$, then

$$\Psi_r(s) = \Phi_r(s) - \binom{m}{r} \Phi_m(s) \qquad (r = 0, 1, \cdots, m)$$

and

$$\Psi_r(s) = \Phi_r(s) = 0 \qquad (r = m + 1, m + 2, \cdots).$$

By (62)

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{[1 - \varphi(s + r\mu)]} \left[\binom{i}{r} + \Phi_{r-1}(s) - \binom{m}{r - 1} \Phi_m(s)\right] \tag{82}$$

for $r = 1, 2, \cdots, m$. The solution of this equation is

$$\Phi_r(s) = \frac{C_r(s)}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j(s)}} \left\{\left[\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=0}^{r} \binom{i}{j} \frac{1}{C_{j-1}(s)}\right] - \left[\sum_{j=0}^{r-1} \binom{m}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=r+1}^{m} \binom{i}{j} \frac{1}{C_{j-1}(s)}\right]\right\} \tag{83}$$

where $C_j(s)$ is defined by (79).

VII. THE LIMITING DISTRIBUTION $\{P_k^*\}$

Now we shall prove

Theorem 5. If $F(x)$ is not a lattice distribution and its mean α is finite, then the limiting distribution

$$\lim_{t \to \infty} P_k(t) = P_k^* \qquad (k = 0, 1, \cdots)$$

exists and is independent of the initial distribution. We have

$$P_{k+1}^* = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots) \tag{84}$$

and

$$P_0^* = 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{k+1}, \tag{85}$$

where $\{P_k\}$ *is defined by* (38).

Proof. By the theory of Markov chains we can conclude that

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{P_k}{\alpha}. \tag{86}$$

Furthermore, it is clear that the difference of the number of transitions $E_k \to E_{k+1}$ and $E_{k+1} \to E_k$ occurring in the time interval $(0,t]$ is at most 1. Accordingly, if we denote by $N_k(t)$ the expectation of the number of transitions $E_{k+1} \to E_k$ occurring in the time interval $(0,t]$, then

$$| p_k M_k(t) - N_k(t) | \leqq 1 \tag{87}$$

for all $t \geqq 0$. Further,

$$N_k(t) = (k+1)\mu \int_0^t P_{k+1}(u)\, du, \tag{88}$$

for, if we consider the process $\{\xi(t)\}$ only at those instants when there is state E_{k+1}, then the transitions $E_{k+1} \to E_k$ form a Poisson process of density $(k+1)\mu$. Thus, by (86), (87), and (88),

$$\lim_{t\to\infty} \frac{(k+1)\mu}{t} \int_0^t P_{k+1}(u)\, du = \lim_{t\to\infty} \frac{N_k(t)}{t} = \lim_{t\to\infty} \frac{p_k M_k(t)}{t} = \frac{p_k P_k}{\alpha};$$

that is,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_{k+1}(u)\, du = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots). \tag{89}$$

If we prove that the limiting distribution

$$\lim_{t\to\infty} P_k(t) = P_k^* \qquad (k = 0, 1, 2, \cdots)$$

exists, then it follows by (89) that

$$P_{k+1}^* = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots), \tag{90}$$

and so

$$P_0^* = 1 - \sum_{k=0}^{\infty} P_{k+1}^* = 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{(k+1)}. \tag{91}$$

To prove the existence of the limiting distribution we need the following auxiliary theorem: If $F(x)$ is not a lattice distribution, then

$$\lim_{t\to\infty} \frac{p_k M_k(t+h) - p_k M_k(t)}{h} \tag{92}$$

exists for every $h > 0$ and is independent of h and the initial state. This is a consequence of a theorem of Blackwell.[14] For the time differences between successive transitions $E_k \to E_{k+1}$ are identically distributed, independent, positive random variables, and, if $F(x)$ is not a lattice distribution, then these random variables have no lattice distribution either. If (92) exists, then it follows that

$$\lim_{t\to\infty} \frac{M_k(t+h) - M_k(t)}{h} = \lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{P_k}{\alpha} \qquad (k = 0, 1, 2, \cdots). \tag{93}$$

Now, by the theorem of total probability, we can write

$$\begin{aligned} P_k(t) = \binom{i}{k} e^{-k\mu x} (1 - e^{-\mu x})^{i-k} [1 - F(t)] \\ + \sum_{j=k-1}^{\infty} \int_0^t \pi_{jk}(t-u)[1 - F(t-u)]\, dM_j(u), \end{aligned} \tag{94}$$

where $\pi_{jk}(t)$ is defined by (18) and it is supposed that $\xi(0) = i$. The event $\xi(t) = k$ may occur in several mutually exclusive ways: there is no call in the time interval $(0,t]$ and, with the exception of k, all the i connections terminate by t; or the last call in the time interval $(0,t]$ is the nth $(n = 1, 2, \cdots)$ one and it finds state E_j $(j = k-1, k, \cdots)$. If $\tau_n = u$ $(0 < u \leqq t)$, then during the time interval $(u,t]$ no new call arrives [the probability of which is $1 - F(t-u)$] and with the exception of k connections every connection terminates by t [the probability of which is $\pi_{jk}(t-u)$].

Applying Blackwell's theorem to (94) and using $\alpha < \infty$, it follows that

$$\lim_{t\to\infty} P_k(t) = P_k^* \qquad (k = 0, 1, \cdots)$$

exists and

$$P_k^* = \sum_{j=k-1}^{\infty} p_{jk}^* P_j, \tag{95}$$

where

$$p_{jk}^* = \frac{1}{\alpha} \int_0^\infty \pi_{jk}(x)[1 - F(x)]\, dx. \tag{96}$$

It is easy to see from (95) that $\{P_k^*\}$ is a probability distribution.

VIII. THE DETERMINATION OF $\gamma_k(s)$

Define

$$\gamma_k(s) = \int_0^\infty e^{-sx}\, dG_k(x) = \frac{D_k(s)}{D_{k+1}(s)}, \tag{97}$$

where $D_0(s) = 1$. We are going to determine $D_r(s) (r = 1, 2, \cdots)$.

Write $D_r(s)$ in the following form:

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s), \tag{98}$$

where $\Delta^j D_0(s)$ is the jth difference of $D_r(s)$ at $r = 0$; that is,

$$\Delta^j D_0(s) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} D_i(s). \tag{99}$$

Then $D_r(s)$ is uniquely determined by its differences.

Now we shall prove

Theorem 6. Starting from $D_0(s) = \Delta^0 D_0(s) = 1$, *the functions* $D_r(s)$ $(r = 0, 1, 2, \cdots)$ *and the differences* $\Delta^j D_0(s)$ $(j = 0, 1, 2, \cdots)$ *can be obtained successively by the recurrence formulas*

$$\begin{aligned} &\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D_j(s) \\ &\qquad = \varphi(s + j\mu) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} [p_j D_{j+1}(s) + q_j D_j(s)] \end{aligned} \tag{100}$$

and

$$\Delta^j D_0(s) = \frac{\varphi(s + j\mu)}{1 - \varphi(s + j\mu)} \sum_{i=0}^{j} \binom{j}{i} (\Delta^{j-i} p_i) \Delta^{i+1} D_0(s) \tag{101}$$

respectively. Here

$$\Delta^{j-i} p_i = \sum_{\nu=0}^{j-i} (-1)^\nu \binom{j - i}{\nu} p_{j-\nu}. \tag{102}$$

Proof. By the theorem of total probability we can write for $r = 0, 1, 2, \cdots$ that

$$G_r(x) = \int_0^x \sum_{j=0}^{r} \binom{r}{j} e^{-j\mu y} (1 - e^{-\mu y})^{r-j} \cdot [p_j G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j G_j(x - y) * \cdots * G_r(x - y)] \, dF(y), \tag{103}$$

where the empty convolution product is equal to 1. Let us consider the instant of a transition $E_{r-1} \to E_r$ and measure time from this instant. Then $G_r(x)$ is the probability that the next transition $E_r \to E_{r+1}$ occurs in the time interval $(0,x]$. This event may occur in the following mutually exclusive ways: the first call in the time interval $(0,x]$ arrives at the instant $y(0 < y \leqq x)$, it finds state E_j $(j = 0, 1, \cdots, r)$, the probability of which is

$$\binom{r}{j} e^{-j\mu y}(1 - e^{-\mu y})^{r-j},$$

and, in the time interval $(y,x]$, a transition $E_r \to E_{r+1}$ occurs, the probability of which is

$$p_j G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j G_j(x - y) * \cdots * G_r(x - y).$$

Introduce the notation

$$q_{r,j}(s) = \binom{r}{j} \int_0^\infty e^{-sx} e^{-j\mu x}(1 - e^{-\mu x})^{r-j} \, dF(x) \tag{104}$$

and form the Laplace-Stieltjes transform of (103); then

$$\gamma_r(s) = \sum_{j=0}^{r} q_{rj}(s) \left[p_j \prod_{i=j+1}^{r} \gamma_i(s) + q_j \prod_{i=j}^{r} \gamma_i(s) \right] \qquad (r = 0, 1, 2, \cdots),$$

where the empty product is 1. Now using (97) we find

$$D_r(s) = \sum_{j=0}^{r} q_{rj}(s)[p_j D_{j+1}(s) + q_j D_j(s)] \qquad (r = 0, 1, 2, \cdots). \tag{105}$$

This is already a recurrence formula for the determination of $D_r(s)$ $(r = 0, 1, 2, \cdots)$, but the coefficients can be simplified further.

If we form

$$\Delta^j D_0(s) = \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} D_l(s),$$

where $D_l(s)$ is replaced by (105), and take into consideration that

$$\sum_{l=i}^{j} (-1)^{j-l} \binom{j}{l} q_{l,i}(s) = (-1)^{j-i} \binom{j}{i} \varphi(s + j\mu), \tag{106}$$

then we obtain

$$\Delta^j D_0(s) = \varphi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [p_i D_{i+1}(s) + q_i D_i(s)]. \tag{107}$$

Now, comparing (99) and (107), we obtain (100).

On the other hand, by (107) it follows that

$$\Delta^j D_0(s) = \varphi(s + j\mu)\Delta^j D_0(s) + \varphi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} p_i \Delta D_i(s),$$

whence

$$\Delta^j D_0(s) = \frac{\varphi(s + j\mu)}{1 - \varphi(s + j\mu)} \Delta^j [p_0 \Delta D_0(s)] \tag{108}$$

and here

$$\Delta^j [p_0 \Delta D_0(s)] = \sum_{i=0}^{j} \binom{j}{i} (\Delta^{j-i} p_i) \Delta^{i+1} D_0(s), \tag{109}$$

where

$$\Delta^{j-i} p_i = \sum_{\nu=0}^{j-i} (-1)^{\nu} \binom{j-i}{\nu} p_{j-\nu}. \tag{110}$$

This proves (101).

Example 10. In the infinite line case, i.e., when $p_j = 1$ ($j = 0, 1, 2, \cdots$), (101) has the following simple form:

$$\Delta^{j+1} D_0(s) = \frac{1 - \varphi(s + j\mu)}{\varphi(s + j\mu)} \Delta^j D_0(s) \qquad (j = 0, 1, 2, \cdots), \tag{111}$$

whence

$$\Delta^j D_0(s) = \prod_{i=0}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right] \tag{112}$$

and

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \left(\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right). \tag{113}$$

Example 11. If $p_0 = 1$ and $p_j = p$ $(j = 1, 2, \cdots)$, then (101) reduces to the following difference equation:

$$\Delta^{j+1}D_0(s) - \frac{1 - \varphi(s + j\mu)}{\varphi(s + j\mu)} \Delta^j D_0(s) + (-1)^j \frac{q[1 - \varphi(s)]}{p\varphi(s)} = 0 \qquad (j = 0, 1, 2, \cdots). \tag{114}$$

A simple calculation shows that the solution of (114) is

$$\Delta^j D_0(s) = \left\{ p \prod_{i=0}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right] - \frac{q[1 - \varphi(s)]}{p\varphi(s)} \sum_{\nu=1}^{j-1} (-1)^\nu \prod_{i=\nu+1}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \right\}, \tag{115}$$

and finally,

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s). \tag{116}$$

Theorem 7. Suppose that $\xi(0) = 0$ and under this condition denote by $M_k(t)$ the expectation of the number of calls arriving in the time interval $(0,t]$ which find exactly k lines busy. Let

$$\mu_k(s) = \int_0^\infty e^{-st} \, dM_k(t). \tag{117}$$

Then

$$\rho_k(s) = 1 - \frac{1}{p_k D_{k+1}(s) \mu_k(s)}, \tag{118}$$

where $D_{k+1}(s)$ is given by Theorem 6 and $\mu_k(s)$ is given by

$$\mu_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \Phi_r(s), \tag{119}$$

where $\Phi_r(s)$ can be obtained by Lemma 1.

Proof. The expected number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0,t]$ is evidently $p_k M_k(t)$. The time differences between consecutive transitions $E_k \to E_{k+1}$ are identically distributed, independent random variables with distribution function $R_k(x)$. By using renewal theory we can write that

$$p_k M_k(t) = G_0(t) * G_1(t) * \cdots * G_k(t) * [I(t) + R_k(t) + R_k(t) * R_k(t) + \cdots], \tag{120}$$

where $I(t) = 1$ if $t \geqq 0$ and $I(t) = 0$ if $t < 0$. Forming the Laplace-

Stieltjes transform of (121), we obtain

$$p_k\mu_k(s) = \frac{\gamma_0(s)\gamma_1(s)\cdots\gamma_k(s)}{1 - \rho_k(s)} = \frac{1}{D_{k+1}(s)[1 - \rho_k(s)]}, \tag{121}$$

whence (118) follows.

Since we know the distribution functions $G_k(x)$ and $R_k(x)$ ($k = 0, 1, 2, \cdots$), the distribution of the number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0,t]$ can be obtained easily.

IX. THE OVERFLOW TRAFFIC

Suppose that $p_j = 1$ ($j = 0, 1, 2, \cdots$) and that the telephone lines are numbered by 1, 2, 3, $\cdots$. Further suppose that an incoming call realizes a connection through the idle line that has the lowest serial number. Consider the group $(1, 2, \cdots, m)$. Denote by $\pi_m^{(n)}$ the probability that the nth call finds every line busy in the group $(1, 2, \cdots, m)$. The distances between successive calls which find every line busy in the group $(1, 2, \cdots, m)$ are evidently identically distributed, independent random variables with distribution function, say, $G_m(x)$.

Palm[2] proved that

$$\pi_m = \lim_{n\to\infty} \pi_m^{(n)} = \frac{1}{\sum_{r=0}^{m} \binom{m}{r} \frac{1}{C_r}}, \tag{122}$$

where C_r is defined by (3). This is in agreement with (6). In this case it is easy to see that $\pi_m^{(n)} = P_m^{(n)}$, where the distribution $\{P_k^{(n)}\}$ is defined in Example 2 of Section IV.

In Refs. 10 and 11 it is shown that

$$\int_0^\infty e^{-sx}\, dG_m(x) = \frac{\sum_{r=0}^{m} \binom{m}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)}\right]}{\sum_{r=0}^{m+1} \binom{m+1}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)}\right]}, \tag{123}$$

where the empty product means 1. It is easy to see that $G_m(x)$ agrees with the corresponding $G_m(x)$ defined in Section VIII when $p_j = 1$ ($j = 0, 1, 2, \cdots$). Thus (123) can be obtained from (97) and (113).

Remark 5. Denote by Γ_m the expectation of the random variable which is the difference of call numbers of successive calls, both of which find all lines busy in the group $(1, 2, \cdots, m)$. Knowing Γ_m, we can write that

$$\pi_m = \lim_{n\to\infty} \pi_m^{(n)} = \frac{1}{\Gamma_m} \tag{124}$$

and

$$\int_0^\infty x\, dG_m(x) = \alpha\Gamma_m. \tag{125}$$

In Ref. 6 it is shown that Γ_r $(r = 1, 2, \cdots)$ satisfies the following recurrence formula:

$$\begin{aligned}\Gamma_r = q_{r,0}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_r) + q_{r,1}(\Gamma_2 + \Gamma_3 + \cdots + \Gamma_r) \\ + \cdots + q_{r,r-2}(\Gamma_{r-1} + \Gamma_r) + q_{r,r-1}\Gamma_r + 1,\end{aligned} \tag{126}$$

where

$$q_{r,j} = \binom{r}{j}\int_0^\infty e^{-j\mu x}(1 - e^{-\mu x})^{r-j}\, dF(x) \qquad (j = 0, 1, \cdots, r). \tag{127}$$

The solution of (126) is given by

$$\Gamma_r = \sum_{j=0}^{r}\binom{r}{j}\prod_{i=1}^{j}\left(\frac{1 - \varphi_i}{\varphi_i}\right) \qquad (r = 1, 2, \cdots). \tag{128}$$

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